Indirect control with quantum accessor (I): coherent control of multi-level system via qubit chain

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A general protocol for controlling any finite dimensional quantum system (N-dimensional qudit) through a quantum accessor is proposed with a built-in feedback mechanism. As an intermediate system that can be well controlled directly, the accessor consists of a number of coupled qubits. The complete controllability of such indirectly controlled system is investigated in detail. The general approach is applied to the indirect control of two and three dimensional quantum systems. For two and three dimensional systems simpler indirect control scheme is also presented.

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I. INTRODUCTION

Quantum control is essentially understood as a coherence preserving manipulation of a quantum system, which enables a time evolution from an arbitrary initial state to an arbitrarily given target state [1, 2, 3, 4]. Recently quantum control has attracted much attention due to its intrinsic relation to quantum information processing algorithms [5]. It has been demonstrated that the universality of quantum logic gates can be well understood from the viewpoint of quantum controllability [6], and the tools of quantum coherent control may be used to design protocols of quantum computing [7].

In connection with the fundamental limit of quantum information processing in physics, we has developed an indirect scheme for quantum control [8] where the *controller* is a quantum system and the operations of quantum control are determined by the initial state of the quantum *controller*. This scheme includes a built-in feedback mechanism, which enables the quantum *controller* to probe the status of the controlled system and then to manipulate its instantaneous time evolution in a unified coherent process. However, due to the quantum decoherence induced by the quantum control itself, the quantum controllability is limited by some uncertainty relation in a well-armed quantum control process. Here the point is, we notice, that the controller itself needs to be well controlled for the exact preparation of a proper initial state. The protocol to be developed in this paper is based on this observation. In this protocol, which may well be called a protocol of indirect control, the "quantized controller" (or quantum accessor) interacts with the controlled system coherently, and a classical external field couples with the quantum accessor to prepare the initial state we need. From physical point of view indirect control is undoubtedly meaningful. Actually, in many physical situations it is very difficult to control the state of quantum system directly, but it is easy to manipulate the state of quantum accessor and thus the state of the *system* via their fixed interaction.

About quantum controllability, recently there has been some powerful conclusions [9], and some well established examples [10, 11]. From these results we observe that it is not difficult to design a quantum accessor which can be well controlled to arrive at an expected initial state. In fact, for the simple case where both the controlled system and the quantum accessor are spin-1/2 particles, the controllability problem has been well investigated most recently [12, 13] in the spirit of Refs. [14, 15], which consider quantum controllability in connection with quantum measurement.

In this paper we will consider an N-energy level system $\mathcal{S}$, and will study the controllability of the total system formed by the controlled quantum system $\mathcal{S}$ plus a quantum accessor $\mathcal{A}$. In the theoretic framework of quantum control, it is assumed that the time evolution of the total system of $\mathcal{S}$ and $\mathcal{A}$ with Hamiltonian $H_0 = H_S + H_A + H_{SA}$ can be externally modified by a family of additional steering fields $\{u_j(t)\}$ in a suitable parameter space through the control Hamiltonian

$$H_c = \sum_j u_j(t) W_j(a, s). \tag{1}$$

Here $H_S = H_S(s)$ ($H_A = H_A(a)$) is the free Hamiltonian of $\mathcal{S}$ ($\mathcal{A}$) of variable s (a) defined on the Hilbert space V_S (V_A) and the coupling Hamiltonian $H_{SA} = H_{SA}(s, a)$ between the system $\mathcal{S}$ and the accessor $\mathcal{A}$ is generally defined on the space $V_S \otimes V_A$. The control operators $W_j(a, s)$ are usually defined also in $V_S \otimes V_A$.

Obviously it is rather trivial to consider the controllability problem of the total system of $\mathcal{S}$ and $\mathcal{A}$ when $W_j(a, s)$ depends on both s and a since this is essentially the conventional classical control problem of the composite quantum system of $\mathcal{S}$ and $\mathcal{A}$. But it is equally obvious that an important situation will arise if $W_j(a, s)$ is con-

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FIG. 1: Illustration of indirect quantum control:(a) An external field classically manipulates the quantum accessor and then indirectly controls the quantum system coupling to the accessor with a fixed interaction. (b) When each state in the total Hilbert space $V_S \otimes V_A$ is reachable under the control via the external classical field only acting on the accessor, each state in the Hilbert space V_S must be reachable. This enables a complete controllability for the indirect control of the controlled system

strained to the space of accessor, namely, $\partial_s W_j(a,s) = 0$ or $W_j(a,s) = W_j(a)$. This case is not at all trivial: it suggests the possibility of controlling the quantum system $\mathcal{S}$ through the control of the variables of the quantum accessor. In fact, this probability is exactly what we will probe in this paper.

We will prove that under some general conditions the control of $\mathcal{A}$ variables can indeed result in a complete quantum control of the whole system and thus lead to an ideal control of its subsystem, the original controlled quantum system $\mathcal{S}$. From mathematical point of view, if the whole system is ergodic in the whole Hilbert space $V_S \otimes V_A$, then each state in the subspace V_S must be reachable by the subsystem $\mathcal{S}$ in the same control process. Here we should point out that a broad dynamical-algebraic framework has been presented, from different motivations and approaches, for analyzing the quantum control properties in terms of the group representation theory [16, 17].

The present paper is the first one of our series papers on indirect quantum control. In the following discussion, we take an arbitrary N-energy level system (the qubit) as the controlled system $\mathcal{S}$ and the spin-1/2 chain with Ising type coupling as the quantum accessor. The controlled system $\mathcal{S}$ and the accessor $\mathcal{A}$ is coupled constantly. In the terminology of group theory, this quantum control problem is cased to the Lie group structure [18, 19]

$$U(N)_S \otimes G_A = U(N)_S \otimes U(2)_1 \otimes ... \otimes U(2)_M. \qquad (2)$$

The remaining part of this paper is organized as follows. In section II, we model the controlled system $\mathcal{S}$ and the accessor $\mathcal{A}$, and formulate the indirect control system. In Section III, we systematically investigate the conditions concerning the complete controllability of the indirect control system, including the coupling between the system and the intermediate system. In Sections IV and V, we apply the general approach to two and three dimensional cases, respectively. Besides, for the two and three dimensional systems, we will discuss more economical indirect control. Finally, we make a short summary and some remarks in Section VI.

II. INDIRECT QUANTUM CONTROL WITH MULTI-QUBIT ENCODING

First of all, let us point out that throughout this paper the symbol i stands for the complex number $\sqrt{-1}$.

Let $\mathcal{S}$ be the N- level quantum system (or qudit), described by the Hamiltonian

$$H_S = \sum_{j=1}^{N} E_j e_{jj}. \qquad (3)$$

Here E_j is the eigen energy and e_{jk} stands for the $N \times N$ matrix with the entries $(e_{jk})_{lm} = \delta_{jl}\delta_{km}$. Without losing generality, we suppose that the Hamiltonian H_S is traceless, namely $\mathrm{tr} H_S = 0$ or $\sum_{j=1}^{N} E_j = 0$. Our aim is to answer the question: can we steer the system $\mathcal{S}$ from an initial state to a target state through an intermediate quantum system, the accessor $\mathcal{A}$ and a family of classical fields which control the accessor $\mathcal{A}$ only?

Intuitively, we need a high dimensional accessor $\mathcal{A}$ to control a high dimensional controlled system. We will use a qubit chain to implement this high dimensional accessor $\mathcal{A}$. Suppose that $\mathcal{A}$ consists of M qubits coupled through nearest neighbor interaction with the Hamiltonian $H_A = H_A^0 + H_A'$:

$$H_A^0 = \sum_{j=1}^{M} \hbar\omega_j \sigma_z^j, \quad H_A' = \sum_{j=1}^{M-1} c_j \sigma_x^j \sigma_x^{j+1}, \qquad (4)$$

where $c_j \neq 0$ is the coupling constant of the nearest neighbor interaction of qubits, $2\hbar\omega_j$ is the level spacing of the j-th qubit, and σ_α^j ($\alpha = x, y, z$; $j = 1, 2, \cdots, M$) is the Pauli's matrix σ_α of the j-th qubit

$$\sigma_\alpha^j = 1 \otimes \cdots \otimes 1 \otimes \sigma_\alpha \otimes 1 \otimes \cdots \otimes 1. \qquad (5)$$

The Hamiltonian (4) describes the well known Ising model and can be used to simulate a quantum computer by appropriate coding [25]

To control the system $\mathcal{S}$ through $\mathcal{A}$, $\mathcal{S}$ has to be coupled to $\mathcal{A}$. We first excite the system $\mathcal{S}$ by applying a *constant* classical field on the system $\mathcal{S}$ via the dipole interaction

$$H_S' = \sum_{j=1}^{N-1} d_j x_j \otimes 1_A, \tag{6}$$

where d_j's are time-independent real coupling constants, and x_j's are the Hermitian operators defined as $x_j = e_{j,j+1} + e_{j+1,j}$. For later use we define x_{jk}, y_{jk} $(1 \le j < k \le N)$and h_j as follows:

$$\begin{aligned} x_{jk} &= e_{jk} + e_{kj}, \\ y_{jk} &= i(e_{jk} - e_{kj}), \\ h_j &= e_{j,j} - e_{j+1,j+1}. \end{aligned} \tag{7}$$

Notice that $x_j = x_{j,j+1}$ by definition. For this reason, let us define $y_j = y_{j,j+1}$. We remark here that with the fixed couplings of $\mathcal{S}$ to an external field, the Hamiltonian of $\mathcal{S}$ can still be diagonalized to take the same form as that of H_S, but the interaction (4) between $\mathcal{S}$ and $\mathcal{A}$ will then have a complicated form. The skew-Hermitian operators ix_{jk}, iy_{jk} and ih_j $(1 \le j < k \le N)$constitute the well-known Chevalley basis of the Lie algebra su(N) [18]. Hereafter we use 1_S and 1_A to denote the identity operator on the Hilbert spaces of the system and the accessor respectively. We note that, different from the conventional control problem, here the interaction H_S' is time-independent.

In the following discussion, for convenience for $\alpha_j \in \{x, y, z, 0\}$, $j = 1, 2, \cdots, M$, we use the abbreviation

$$[\alpha] = (\alpha_1, \alpha_2, \cdots, \alpha_M),$$

and define

$$\sigma_{[\alpha]} = \prod_{j=1}^{M} \sigma_{\alpha_j}^j, \, \sigma_0 = 1.$$

The coupling between the system $\mathcal{S}$ and the accessor $\mathcal{A}$ is generally given as

$$H_{SA} = \sum_{j=1}^{N-1} \sum_{k=1}^{2} \sum_{[\alpha]} g_{[\alpha]}^{j(k)} s_j^{(k)} \otimes \sigma_{[\alpha]}, \tag{8}$$

where in the summation over $[\alpha]$ each α_j is restricted to the set $\{x, y\}$, $s_j^{(k)}$ $(1 \le j \le N-1,\ k = 1, 2)$ denotes either x_j or y_j defined in Eq. (7):

$$s_j^{(k)} = \begin{cases} x_j, & \text{when } k = 1, \\ y_j, & \text{when } k = 2, \end{cases} \tag{9}$$

and $g_{[\alpha]}^{j(k)}$ is the coupling constant. The above coupling is general for spin-large spin interaction and reduces to the Heisenberg type coupling when $N = 2$.

Then the total system of $\mathcal{S}$ and $\mathcal{A}$ is described by the Hamiltonian H_0

$$H_0 = H_S \otimes 1_A + H_S' + 1_S \otimes H_A + H_{SA}. \tag{10}$$

The central point of our protocol is to control the system $\mathcal{S}$ *indirectly* by controlling the accessor $\mathcal{A}$ using classical fields. Suppose we can completely control every qubit using two independent external fields $f_j(t)$ and $f_j'(t)$, $j = 1, 2, \cdots, M$, which couple to a qubit in the following way [10, 11]:

$$H_x^j = 1_S \otimes \sigma_x^j, \tag{11}$$

$$H_y^j = 1_S \otimes \sigma_y^j. \tag{12}$$

Then the total Hamiltonian for the indirect control is obtained as

$$H = H_0 + \sum_{j=1}^{M} \left(f_j(t) H_x^j + f_j'(t) H_y^j \right). \tag{13}$$

In this paper we shall examine under what conditions the control system (13) is completely controllable.

III. COMPLETE CONTROLLABILITY OF INDIRECT CONTROL

In this section we consider the complete controllability of the system $\mathcal{S}$: whether the system $\mathcal{S}$ can be controlled completely by controlling the accessor $\mathcal{A}$. For this purpose, it is enough to investigate whether the Lie algebra $\mathcal{L}$ generated by iH_0, iH_x^j and iH_y^j is $su(2^M N)$, which generates the Lie group of all the unitary operations on $V_S \otimes V_A$ through the single parameter subgroups. If $\mathcal{L}$ is equal to $su(2^M N)$, the system is completely controllable. Otherwise, the system is partly controllable.

For the skew-hermitian operators

$$iH_0, \quad iH_x^j, \quad iH_y^j, \ \ j = 1, 2, ..., M \tag{14}$$

to generate the Lie algebra $su(2^M N)$ some conditions should be satisfied. This section is mainly devoted to the investigation of such conditions when M is greater than 2, the cases with $M = 1, 2$ being left to the subsequent sections.

For convenience, we introduce the following notions about conditions on the system $\mathcal{S}$:

Condition 1. $c_j \neq 0$ for $j = 1, 2, \cdots, M-1$;

Condition 2. There exist $2(N-1) \equiv N'$ elements $[\beta]_1, [\beta]_2, \cdots, [\beta]_{N'}$ of the set $\{x, y\}^M$ such that the matrix

$$G = \begin{bmatrix} g_{[\beta]_1}^{1(1)} & \cdots & g_{[\beta]_1}^{(N-1)(1)} & g_{[\beta]_1}^{1(2)} & \cdots & g_{[\beta]_1}^{(N-1)(2)} \\ g_{[\beta]_2}^{1(1)} & \cdots & g_{[\beta]_2}^{(N-1)(1)} & g_{[\beta]_2}^{1(2)} & \cdots & g_{[\beta]_2}^{(N-1)(2)} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ g_{[\beta]_{N'}}^{1(1)} & \cdots & g_{[\beta]_{N'}}^{(N-1)(1)} & g_{[\beta]_{N'}}^{1(2)} & \cdots & g_{[\beta]_{N'}}^{(N-1)(2)} \end{bmatrix} \tag{15}$$

is not singular, namely, the determinant of G is nonzero;

Condition 3. The complete controllability conditions on the coupling constants and the eigen-energy E_j, presented in Ref.[10,11].

Notice that Condition 2 implies the restriction $2^M \ge 2(N-1)$.

Lemma 1 *Given an arbitrary* $[\beta]=(\beta_1,\beta_2,\cdots,\beta_M)\in\{x,y\}^M$*, we have*

$$i^M\left[1_S\otimes\sigma^M_{\beta_M},\left[1_S\otimes\sigma^{M-1}_{\beta_{M-1}},\left[\cdots,\left[1_S\otimes\sigma^1_{\beta_1},i(1_S\otimes H'_A)\right]\cdots\right]\right]\right]$$
$$=\begin{cases}4ic_1\delta_{\beta_1 y}\delta_{\beta_2 y}(1_S\otimes\sigma^1_z\sigma^2_z), & \textit{when } M=2;\\ 0 & \textit{when } M>2.\end{cases}\quad(16)$$

This lemma can be verified directly. We would rather omit the proof.

Lemma 2 *If* $i(1_S\otimes\sigma^j_x\sigma^{j+1}_x)\in\mathcal{L}$ $(j=1,2,\cdots,M-1)$*, then for an arbitrary* $[\alpha]\in\{x,y,z,0\}^M$ *except* $[\alpha]=(0,0,\cdots,0)$ *we have* $i(1_S\otimes\sigma_{[\alpha]})\in\mathcal{L}$.

Proof. We first consider the element $i(1_S\otimes\sigma_{[\alpha]})$ with $\alpha_1=\alpha_2=\cdots=\alpha_M=x$. From (11) and (12) we have $1_S\otimes\sigma^j_y\in\mathcal{L}$ and

$$-2^{-1}\left[iH^j_x,iH^j_y\right]=i(1_S\otimes\sigma^j_z)\in\mathcal{L}.\quad(17)$$

As a result,

$$2^{-1}[i(1_S\otimes\sigma^2_x\sigma^3_x),\ i(1_S\otimes\sigma^2_z)]=i(1_S\otimes\sigma^2_y\sigma^3_x)\in\mathcal{L},$$
$$-2^{-1}[i(1_S\otimes\sigma^1_x\sigma^2_x),\ i(1_S\otimes\sigma^2_y\sigma^3_x)]=i(1_S\otimes\sigma^1_x\sigma^2_z\sigma^3_x)\in\mathcal{L},$$
$$2^{-1}[i(1_S\otimes\sigma^1_x\sigma^2_z\sigma^3_x),i(1_S\otimes\sigma^2_y)]=i(1_S\otimes\sigma^1_x\sigma^2_x\sigma^3_x)\in\mathcal{L}.$$

In the same way we can obtain $i(1_S\otimes\sigma^1_x\sigma^2_x\sigma^3_x\sigma^4_x)\in\mathcal{L}$. Now we easily observe that by repeating this procedure we can prove that

$$i(1_S\otimes\sigma^1_x\sigma^2_x\cdots\sigma^M_x)\in\mathcal{L}.\quad(18)$$

Next, we consider the elements $i(1_S\otimes\sigma_{[\alpha]})$ with $\alpha_j\in\{x,y,z\}$. It is easy to see that such elements lie in the Lie algebra generated by $\{i(1_S\otimes\sigma^1_x\sigma^2_x\cdots\sigma^M_x),iH^j_x,iH^j_y|j=1,2,\cdots,M\}$, which is a subset of $\mathcal{L}$. It then follows that $i(1_S\otimes\sigma_{[\alpha]})\in\mathcal{L}$ for $\alpha_j=x,y,z$.

Finally, we deal with the general element $i(1_S\otimes\sigma_{[\alpha]})$It remains to prove that $i(1_S\otimes\sigma_{[\alpha]})\in\mathcal{L}$ for the α with some $\alpha'_j s$ being zero. To this end, we observe that

$$2^{-1}[i(1_S\otimes\sigma^1_x\sigma^2_x),\ i(1_S\otimes\sigma^2_z)]=i(1_S\otimes\sigma^1_x\sigma^2_y)\in\mathcal{L},$$

so it follows that

$$-2^{-1}\left[i(1_S\otimes\sigma^1_x\sigma^2_x\cdots\sigma^M_x),i(1_S\otimes\sigma^1_x\sigma^2_y)\right]$$
$$=i(1_S\otimes\sigma^1_0\sigma^2_z\sigma^3_x\cdots\sigma^M_x)\in\mathcal{L}.$$

Now having this element at our disposal, with the help of iH^j_x and iH^j_y we can generate in $\mathcal{L}$ all the elements $i(1_S\otimes\sigma_{[\alpha]})$ with $\alpha_1=0$ and $\alpha_j\in\{x,y,z\}$, $j\neq 1$. After a moment's thought, one can see that using this trick we can actually prove that $i(1_S\otimes\sigma_{[\alpha]})\in\mathcal{L}$ for the α with one α_j, not necessarily α_1, being zero. Finally, along the same way we can proceed further to show that $i(1_S\otimes\sigma_{[\alpha]})\in\mathcal{L}$ for the α with n α_j's $(1\le n<M)$ being zero. The lemma is thus proved.

Lemma 3 *When* $M>2$*, if Condition 2 is satisfied, then for* $j=1,2,\cdots,N-1$ *and* $[\alpha]\neq(0,0,\cdots,0)$ *the elements* $ix_j\otimes\sigma_{[\alpha]},iy_j\otimes\sigma_{[\alpha]},ih_j\otimes 1_A$ *lie in* $\mathcal{L}$.

Proof. We have already known that the elements $i(1_S\otimes\sigma^j_z)(j=1,2,\cdots,M)$are contained in $\mathcal{L}$. So $i(1_S\otimes H^0_I)$, which is a linear combination of these elements, is also contained in $\mathcal{L}$. It then follows that $iH_0-i(1_S\otimes H^0_I)\in\mathcal{L}$, namely,

$$iH'_0\equiv iH_S\otimes 1_A+iH'_S+i(1_S\otimes H'_A)+iH_{SA}\in\mathcal{L}.\quad(19)$$

Now for $\beta_j\in\{x,y\}$, let us consider the element

$$i^M\left[1_S\otimes\sigma^M_{\beta_M},\left[1_S\otimes\sigma^{M-1}_{\beta_{M-1}},\left[\cdots,\left[1_S\otimes\sigma^1_{\beta_1},iH'_0\right]\cdots\right]\right]\right],\quad(20)$$

which belongs to $\mathcal{L}$ as $i(1_S\otimes\sigma^j_{\beta_j})$ belongs to $\mathcal{L}$ by definition.

Clearly, the term $i(H_S\otimes 1_A)+iH'_S$ in iH'_0 has no nonzero contribution to this element. Moreover, since $M>2$ Lemma 1 tells us that the term $i(1_S\otimes H'_A)$ has no nonzero contribution either.

By straightforward calculation it then follows that

$$i^M\left[1\otimes\sigma^M_{\beta_M},\left[1\otimes\sigma^{M-1}_{\beta_{M-1}},\left[\cdots,\left[1\otimes\sigma^1_{\beta_1},iH_{SA}\right]\cdots\right]\right]\right]$$
$$=i(-1)^{M+\Delta}2^M\left[\sum_{j=1}^{N-1}\sum_{k=1}^{2}g^{j(k)}_{[\bar\beta]}s^{(k)}_j\right]\otimes\sigma^1_z\cdots\ \sigma^M_z\in\mathcal{L},$$

where $\bar\beta$ is defined as

$$\bar\beta_j=\begin{cases}x, & \text{if } \beta_j=y,\\ y, & \text{if } \beta_j=x,\end{cases}\quad(21)$$

and Δ is the number of y in $\{\beta_j|j=1,2,\cdots,M\}$. Consequently, for each $[\beta]\in\{x,y\}^M$ we have

$$i\left[\sum_{j=1}^{N-1}\sum_{k=1}^{2}g^{j(k)}_{[\bar\beta]}s^{(k)}_j\right]\otimes(\sigma^1_z\cdots\sigma^M_z)\in\mathcal{L}.\quad(22)$$

There are altogether 2^M such elements. Now Condition 2 guarantees that from these elements we can choose $2(N-1)$ linearly independent ones. Then from these linearly independent elements in $\mathcal{L}$ we can derive

$$is^{(k)}_j\otimes(\sigma^1_z\cdots\sigma^M_z)\in\mathcal{L},\ j=1,2,\cdots,N-1,\ k=1,2\quad(23)$$

by the standard method of linear algebra. Using the same method as that in the proof of Lemma 2, we can go further to prove that $is^{(k)}_j\otimes\sigma_{[\alpha]}\in\mathcal{L}$, namely, $ix_j\otimes\sigma_{[\alpha]},iy_j\otimes\sigma_{[\alpha]}\in\mathcal{L}$, for $[\alpha]\neq(0,0,\cdots,0)$. Then the lemma follows directly because we have

$$(-2)^{-1}\left[ix_j\otimes\sigma_{[\alpha]},iy_j\otimes\sigma_{[\alpha]}\right]=ih_j\otimes 1_A.$$

Lemma 4 *When $M > 2$, if Condition 1 and Condition 2 are satisfied, then for $[\alpha] \neq (0,0,\cdots,0)$ we have $i1_S \otimes \sigma_{[\alpha]} \in \mathcal{L}$.*

Proof. We observe that it follows from Lemma 2 that $iH_{SA} \in \mathcal{L}$ and $iH_S \otimes 1_A \in \mathcal{L}$. The former is obvious and the latter is due to the fact

$$iH_S = i \sum_{j=1}^{N-1} \left(E_1 + E_2 + \cdots + E_j\right) h_j, \tag{24}$$

Recalling that we also have $iH_A^0 \in \mathcal{L}$, we obtain

$$\begin{aligned} iH_0'' &\equiv i\left(H_0 - H_S \otimes 1_A - H_A - H_{SA}\right) \\ &= i1_S \otimes \sum_{j=1}^{M-1} c_j \sigma_x^j \sigma_x^{j+1} + i \sum_{j=1}^{N-1} d_j x_j \otimes 1_A \in \mathcal{L}. \end{aligned} \tag{25}$$

It then follows that

$$\left[\left[iH_0'', iH_y^1\right], iH_y^1\right] = -i4c_1\left(1_S \otimes \sigma_x^1 \sigma_x^2\right) \in \mathcal{L}, \tag{26}$$

yielding $i\left(1_S \otimes \sigma_x^1 \sigma_x^2\right) \in \mathcal{L}$ thanks to the condition $c_1 \neq 0$. This leads to the result

$$\begin{aligned} &\left[\left[iH_0'' - ic_1 1_S \otimes \sigma_x^1 \sigma_x^2,\, iH_y^2\right],\, iH_y^2\right] \\ &= -i4c_2\left(1_S \otimes \sigma_x^2 \sigma_x^3\right) \in \mathcal{L}, \end{aligned}$$

namely, $i\left(1 \otimes \sigma_x^2 \sigma_x^3\right) \in \mathcal{L}$ since $c_2 \neq 0$. Repeating this process we can finally prove

$$i\left(1_S \otimes \sigma_x^j \sigma_x^{j+1}\right) \in \mathcal{L}, \quad j = 1, 2, \cdots, M-1. \tag{27}$$

Then the lemma follows from Lemma 2.

Theorem 1 *When $M > 2$, if Condition 1, Condition 2 and Condition 3 are satisfied, then we have $\mathcal{L} = su(2^M N)$.*

Proof. First we claim that under the conditions of the theorem, for $j = 1, 2, \cdots, N-1$

$$i\left(x_j \otimes 1_A\right), i\left(y_j \otimes 1_A\right) \in \mathcal{L}. \tag{28}$$

Recall that $iH_S \otimes 1_A \in \mathcal{L}$ and notice that Eq.(27) implies $iH_A' \in \mathcal{L}$, and hence

$$iH_S' = iH_0'' - iH_A' \in \mathcal{L}.$$

Then according to the result of Ref.[10,11], if Condition 3 is satisfied the elements $i\left(x_j \otimes 1_A\right)$ and $i\left(y_j \otimes 1_A\right)$ are contained in the subalgebra of $\mathcal{L}$ generated by $iH_S \otimes 1_A$ and iH_S'. This proves the claim.

Since the elements of the set $\{ix_{jk}, iy_{jk}, ih_j | 1 \leq j < k \leq N\}$ can be generated from the set $\{ix_j, iy_j | j = 1, 2, \cdots, N-1\}$ it follows from Lemma 3, Lemma 4 and (28) that the following elements are in the Lie algebra $\mathcal{L}$

$$\begin{aligned} &ix_{jk} \otimes 1_A, \quad y_{jk} \otimes 1_A, \quad h_j \otimes 1_A, \\ &x_{jk} \otimes \sigma_{[\alpha]}, \quad y_{jk} \otimes \sigma_{[\alpha]}, \quad h_j \otimes \sigma_{[\alpha]}, \\ &1_S \otimes \sigma_{[\alpha]}, \end{aligned}$$

where $[\alpha] \neq (0,0,\cdots,0)$, and $1 \leq j < k \leq N$. It is easily check that these elements are linearly independent and the total number of these elements is

$$\begin{aligned} &(N^2-1) + (N^2-1)(4^M-1) + (4^M-1) \\ &= (2^M N)^2 - 1 = \dim\left(su(2^M N)\right). \end{aligned} \tag{29}$$

This proves the theorem.

Before leaving this section we would like to note that the coupling between the system and the accessor plays an essential role in the indirect control. In the above given H_{SA} there are $2(N-1) \times 2^M$ coupling terms. Actually as far as the controllability is concerned, we have simpler choices of H_{SA}. For example, we can reduce the number of coupling terms to $2(N-1)$, just enough to guarantee the satisfaction of Condition 2.

IV. INDIRECT CONTROL FOR TWO-DIMENSIONAL SYSTEM

In this section we will consider an explicit example, the indirect control of a two-energy level system, to illustrate the general approach given in last section. We also present a simpler indirect control scheme for 2-dimensional system.

The 2-dimensional quantum system can be described by the Hamiltonian

$$H_S = \hbar\omega_S \sigma_z \otimes 1_A, \tag{30}$$

in terms of Pauli's matrices. In this case, it is possible to use just one qubit as the accessor. The Hamiltonian of the entire control system can be written as

$$\begin{aligned} H &= \hbar\omega_S \sigma_z \otimes 1_A + g\sigma_x \otimes 1 + 1_S \otimes \hbar\omega_I \sigma_z \\ &+ g_{xx}\sigma_x \otimes \sigma_x + g_{xy}\sigma_x \otimes \sigma_y \\ &+ g_{yx}\sigma_y \otimes \sigma_x + g_{yy}\sigma_y \otimes \sigma_y \\ &+ f_1(t)\left(1_S \otimes \sigma_x\right) + f_2(t)\left(1_S \otimes \sigma_y\right). \end{aligned} \tag{31}$$

Here we remark that the excitation term $\sigma_x \otimes 1$ can be removed by rotating the controlled system around y-direction so that $\hbar\omega_S \sigma_z \otimes 1_A + g\sigma_x \otimes 1$ becomes $\hbar\omega_S' \sigma_z \otimes 1_A$. As the price paid, the rotated Hamiltonian contains the terms $g_{zx}'\sigma_z \otimes \sigma_x$ and $g_{zy}'\sigma_z \otimes \sigma_y$:

$$\begin{aligned} H &= \hbar\omega_S' \sigma_z \otimes 1_A + 1_S \otimes \hbar\omega_I \sigma_z \\ &+ g_{xx}'\sigma_x \otimes \sigma_x + g_{xy}'\sigma_x \otimes \sigma_y \\ &+ g_{yx}\sigma_y \otimes \sigma_x + g_{yy}\sigma_y \otimes \sigma_y \\ &+ g_{zx}'\sigma_z \otimes \sigma_x + g_{zy}'\sigma_z \otimes \sigma_y \\ &+ f_1(t)\left(1_S \otimes \sigma_x\right) + f_2(t)\left(1_S \otimes \sigma_y\right). \end{aligned} \tag{32}$$

The following theorem is the main result of this section.

Theorem 2 *Suppose that $g_{xy}g_{yx} \neq g_{xx}g_{yy}$. Then the symplectic Lie algebra $sp(4)$ is included in $\mathcal{L}$. Moreover, if $g \neq 0$ is also satisfied, then $\mathcal{L} = su(4)$.*

Proof. We observe that in the present case, Lemma 1 reduces to the trivially true identity since the coupling term in H_A does not appear. On the other hand the assumption $g_{xy}g_{yx} \neq g_{xx}g_{yy}$ simply means that Condition 2 is satisfied. Therefore Lemma 2 is valid. Noticing that,by definition, $x_1 = \sigma_x$, $y_1 = \sigma_y$ and $h_1 = \sigma_z$ with respect to a proper basis when $N = 2$, we conclude, from Lemma 2 and the fact that $\mathcal{L}$ contains the elements $i(1_S \otimes \sigma_x)$ $i(1_S \otimes \sigma_x)$ by definition, that $\mathcal{L}$ contains the following elements:

$$
\begin{aligned}
&i(1_S \otimes \sigma_\alpha), \quad \alpha = x, y, z; \\
&i\sigma_\alpha \otimes \sigma_\beta, \quad \alpha = x, y, \beta = x, y, z; \\
&i(\sigma_z \otimes 1_A)
\end{aligned} \tag{33}
$$

and thus contains the element $g(i\sigma_x \otimes 1_A)$, which is obtained by subtracting from iH_0 all the other terms, which lie in $\mathcal{L}$.

Now we claim that we can choose a basis of sp(4) from those elements in (33). In fact, we have

$$
i\sigma_z \otimes 1 = \left(\begin{array}{cc|cc} \mathrm{i} & & & \\ & \mathrm{-i} & & \\ \hline & & \mathrm{-i} & \\ & & & \mathrm{i} \end{array}\right), i(1 \otimes \sigma_z) = \left(\begin{array}{cc|cc} \mathrm{i} & & & \\ & \mathrm{i} & & \\ \hline & & \mathrm{-i} & \\ & & & \mathrm{-i} \end{array}\right), \tag{34}
$$

$$
i\sigma_x \otimes \sigma_z = \left(\begin{array}{cc|cc} & \mathrm{i} & & \\ \mathrm{i} & & & \\ \hline & & & \mathrm{-i} \\ & & \mathrm{-i} & \end{array}\right), i\sigma_y \otimes \sigma_z = \left(\begin{array}{cc|cc} & 1 & & \\ -1 & & & \\ \hline & & & -1 \\ & & 1 & \end{array}\right), \tag{35}
$$

$$
i\sigma_x \otimes \sigma_x = \left(\begin{array}{cc|cc} & & \mathrm{i} & \\ & & & \mathrm{i} \\ \hline \mathrm{i} & & & \\ & \mathrm{i} & & \end{array}\right), i\sigma_x \otimes \sigma_y = \left(\begin{array}{cc|cc} & & 1 & \\ & & & 1 \\ \hline -1 & & & \\ & -1 & & \end{array}\right), \tag{36}
$$

$$
i\sigma_y \otimes \sigma_x = \left(\begin{array}{cc|cc} & & 1 & \\ & & & -1 \\ \hline -1 & & & \\ & 1 & & \end{array}\right), i\sigma_y \otimes \sigma_y = \left(\begin{array}{cc|cc} & & \mathrm{-i} & \\ & & & \mathrm{i} \\ \hline \mathrm{-i} & & & \\ & \mathrm{i} & & \end{array}\right), \tag{37}
$$

$$
i(1 \otimes \sigma_x) = \left(\begin{array}{cc|cc} & & & \mathrm{i} \\ & & \mathrm{i} & \\ \hline & \mathrm{i} & & \\ \mathrm{i} & & & \end{array}\right), i(1 \otimes \sigma_y) = \left(\begin{array}{cc|cc} & & & 1 \\ & & 1 & \\ \hline & -1 & & \\ -1 & & & \end{array}\right), \tag{38}
$$

with respect to the ordered basis $\{|0\rangle \otimes |0\rangle, |1\rangle \otimes |0\rangle, |1\rangle \otimes |1\rangle, |0\rangle \otimes |1\rangle\}$. It is readily check that these matrices are linearly independent and satisfy the equation

$$
S^t x + xS = 0, \tag{39}
$$

the defining relation of $sp(4)$, where

$$
S = \begin{pmatrix} & I \\ -I & \end{pmatrix} \tag{40}
$$

and I is the 2×2 identity matrix. This proves the claim, and hence the first part of the theorem, as the dimension of $sp(4)$ is 10.

If $g \neq 0$, from $g(i\sigma_x \otimes 1_A) \in \mathcal{L}$ we can derive $i\sigma_x \otimes 1_A \in \mathcal{L}$. It is easily check that this element, together with the elements in (33), can generate 15 linearly independent elements by Lie bracket operations. As the dimension of $su(4)$ is exactly 15 we conclude that $\mathcal{L} = su(4)$. The proof of Theorem 2 is thus completed.

We remark that it is easy to satisfy the condition $g_{xy}g_{yx} \neq g_{xx}g_{yy}$. For example, we can take

$$
g_{xx} = g_{yy} = 0, g_{xy} = g_{yx} \neq 0, \tag{41}
$$

or

$$
g_{xy} = g_{yx} = 0, g_{xx} = g_{yy} \neq 0. \tag{42}
$$

In both cases, there are only two terms in the coupling between the system $\mathcal{S}$ and the accessor $\mathcal{A}$.

Finally, we point out that, by making full use of the property that the square of Pauli's matrices is unity, which is peculiar to the $N = 2$ case, we can manage to control the system completely by means of simpler couplings between the system and the accessor. Let us consider, as an example, the control system

$$
\begin{aligned}
H_0 &= \hbar\omega_S \sigma_z \otimes 1_A + g\sigma_x \otimes 1 \\
&\quad + 1_S \otimes \hbar\omega_I \sigma_z + g_{xx}\sigma_x \otimes \sigma_x, \\
H_c &= f_1(t)\,(1_S \otimes \sigma_x) + f_2(t)\,(1_S \otimes \sigma_y)
\end{aligned} \tag{43}
$$

where $g \neq 0$ and $g_{xx} \neq 0$. Such a control system is essentially different from the system just discussed above as in this case Condition 2 is never satisfied. One can easily check that

$$
\begin{aligned}
&(2g_{xx})^{-1}\left(-[iH_0, i(1 \otimes \sigma_y)] + 2i\hbar\omega_I \otimes \sigma_x\right) \\
&= i\sigma_x \otimes \sigma_z \in \mathcal{L},
\end{aligned} \tag{44}
$$

from which we further have

$$
\begin{aligned}
&-(2\hbar\omega_S)^{-1}\left[iH_0 - \hbar\omega_I 1 \otimes \sigma_z - ig_{xx}\sigma_x\sigma_x,\ i\sigma_x \otimes \sigma_z\right] \\
&= (2\hbar\omega_S)^{-1}\left[\hbar\omega_S\sigma_z \otimes 1_A + g\sigma_x \otimes 1,\ \sigma_x \otimes \sigma_z\right] \\
&= i\sigma_y \otimes \sigma_z \in \mathcal{L}.
\end{aligned} \tag{45}
$$

Now it should not be difficult to proceed further to prove that the two conclusions of Theorem 2 are still valid though the premise is no longer true. We leave the details to interested readers.

V. INDIRECT CONTROL FOR 3-DIMENSIONAL QUANTUM SYSTEM

In this section we discuss the indirect control of 3-dimensional quantum system based on the approach presented in Section III.

Since Theorem 1 is, generally speaking, not valid when $M \leq 2$, we first consider the possibility of using 3 qubits to control the system, namely, we assume that $M = 3$.

Let $[\beta]_1 = (x,x,x)$, $[\beta]_2 = (x,x,y)$, $[\beta]_3 = (x,y,x)$ and $[\beta]_4 = (y,x,x)$. To satisfy Condition 2, we can simply choose $g^{j(k)}_{[\beta]} = 0$ except that

$$g^{1(1)}_{[\beta]_1} = g^{2(1)}_{[\beta]_2} = g^{1(2)}_{[\beta]_3} = g^{2(2)}_{[\beta]_4} = 1, \tag{46}$$

namely,

$$\begin{aligned} H_{SA} &= x_1 \otimes \sigma^1_y \sigma^2_y \sigma^3_y + x_2 \otimes \sigma^1_y \sigma^2_y \sigma^3_x \\ &\quad + y_1 \otimes \sigma^1_y \sigma^2_x \sigma^3_y + y_2 \otimes \sigma^1_x \sigma^2_y \sigma^3_y. \end{aligned} \tag{47}$$

In fact, in such a case, we have

$$\det \begin{bmatrix} g^{1(1)}_{[\beta]_1} & g^{2(1)}_{[\beta]_1} & g^{1(2)}_{[\beta]_1} & g^{2(2)}_{[\beta]_1} \\ g^{1(1)}_{[\beta]_2} & g^{2(1)}_{[\beta]_2} & g^{1(2)}_{[\beta]_2} & g^{2(2)}_{[\beta]_2} \\ g^{1(1)}_{[\beta]_3} & g^{2(1)}_{[\beta]_3} & g^{1(2)}_{[\beta]_3} & g^{2(2)}_{[\beta]_3} \\ g^{1(1)}_{[\beta]_4} & g^{2(1)}_{[\beta]_4} & g^{1(2)}_{[\beta]_4} & g^{2(2)}_{[\beta]_4} \end{bmatrix} = 1 \tag{48}$$

Now assume Condition 1, then Condition 3 is enough to guarantee the complete controllability. In our present case, Condition 3 has a simple form[10, 11]:

$$\Delta^2_{21} \neq \Delta^2_{32} \text{ and } d_1 \neq 0, d_2 \neq 0 \tag{49}$$

or

$$\Delta^2_{21} = \Delta^2_{32} \text{ and } d_1 \neq \pm d_2 \neq 0, \tag{50}$$

where $\Delta_{jk} \equiv E_j - E_k$ $(3 \geq j > k \geq 1)$ is the energy gap.

Now we consider the possibility of using only two qubits to control the 3-dimensional system. As in this case $M = 2$, the general approach developed in Section III cannot be fully applied. However, we have the following conclusion: if we can control not only each qubit, but also their coupling independently, we can indirectly control the 3-dimensional system using two qubits. In fact, if this is the case, we can take the Hamiltonian as

$$\begin{aligned} H &= H_0 + H^1_c + H^2_c + H^{12}_c \\ H_0 &= \sum_{j=1}^{3} \hbar\omega_S e_{jj} \otimes 1_A + (d_1 x_1 + d_2 x_2) \otimes 1_A \\ &\quad + 1_S \otimes \sum_{j=1}^{2} (\hbar\omega_I \sigma^j_z) \\ &\quad + \sum_{j=1}^{2} \sum_{\alpha_1,\alpha_2 = x,y} g^{j(k)}_{\alpha_1\alpha_2} s^{(k)}_j \otimes (\sigma^1_{\alpha_1} \sigma^2_{\alpha_2}) \end{aligned} \tag{51}$$

$$\begin{aligned} H^j_c &= f_j(t) \left(1_S \otimes \sigma^j_x\right) + f'_j(t) \left(1_S \otimes \sigma^j_y\right) \\ H^{12}_c &= f(t) 1_S \otimes \sigma^1_x \sigma^2_x. \end{aligned} \tag{52}$$

Let $\mathcal{L}$ be the Lie algebra generated by the elements

$$iH_0, \quad i\left(1_S \otimes \sigma^j_x\right), \quad i\left(1_S \otimes \sigma^j_y\right), \quad i\left(1 \otimes \sigma^1_x \sigma^2_x\right), \tag{53}$$

where $j = 1, 2$. Then mathematically the complete controllability condition is $\mathcal{L} = su(4)$. Using a method similar to that in Section III we can prove $\mathcal{L} = su(4)$ if the condition(49) or (50), and the condition

$$\det \begin{bmatrix} g^{1(1)}_{xx} & g^{2(1)}_{xx} & g^{1(2)}_{xx} & g^{2(2)}_{xx} \\ g^{1(1)}_{xy} & g^{2(1)}_{xy} & g^{1(2)}_{xy} & g^{2(2)}_{xy} \\ g^{1(1)}_{yx} & g^{2(1)}_{yx} & g^{1(2)}_{yx} & g^{2(2)}_{yx} \\ g^{1(1)}_{yy} & g^{2(1)}_{yy} & g^{1(2)}_{yy} & g^{2(2)}_{yy} \end{bmatrix} \neq 0 \tag{54}$$

are satisfied. We would rather omit the details to avoid redundancy.

Finally, we conclude this section by pointing out that (54) can be satisfied by simply choosing

$$\begin{aligned} H'_{SA} &= x_1 \otimes \sigma^1_x \sigma^2_x + y_1 \otimes \sigma^1_x \sigma^2_y \\ &\quad + x_2 \otimes \sigma^1_y \sigma^2_x + y_2 \otimes \sigma^1_y \sigma^2_y. \end{aligned} \tag{55}$$

VI. CONCLUSION AND REMARKS

In this paper we propose a general scheme to control an arbitrary finite dimensional quantum system through a quantum accessor which consists of a number of coupled qubits. The conditions for complete controllability are investigated in detail both mathematically and physically. The general approach is applied to the indirect control of two and three dimensional quantum systems. We also present indirect control schemes simpler than the general scheme for two and three dimensional systems.

Before concluding this paper we would like to remark that in the conventional investigation on the controllability of quantum systems, the controls are usually classical or semiclassical since the controlling field is described as a time-dependent functions and directly affects the time evolution of the closed or open quantum systems to be controlled [20, 21, 22, 23, 24]. So it might be more appropriate to name those types of control *(semi)classical control of quantum systems*.

In a forthcoming paper as the second one of our series papers about indirect quantum control we will study a control system where the fixed interaction between the controlled system and the accessor is so weak that it can be neglected approximately when the strong field, which controls the accessor, is switched on.

Acknowledgement

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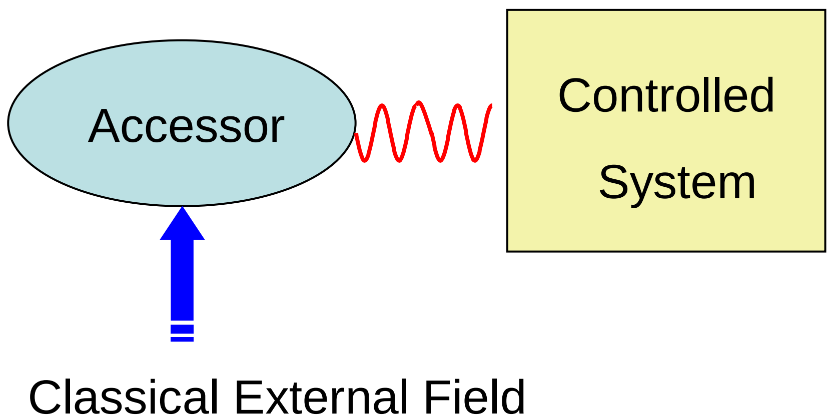
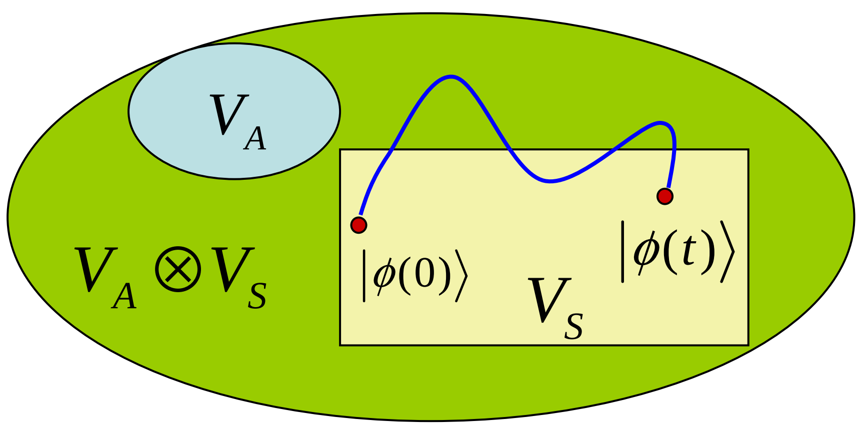

Accessor
Controlled
System
Classical External Field

(a)


V_A
$V_A \otimes V_S$
$|\phi(0)\rangle$
V_S
$|\phi(t)\rangle$

(b)